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Performance of A Nonlinear Dynamical System of Two-Dimensional TPC Decoder

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Abstract **- The nonlinear dynamical system of turbo code decoders and LDPC code decoders has been analyzed in detail by a lot of researchers. But the dynamical behaviours of turbo product code(TPC) decoders have not been reported so far. In this paper, we model a discrete dynamical system of two-dimensional (2-D) TPC decoders using one input parameter (corresponding to the SNR) and one output parameter (corresponding to E(*l*)-the mean square value of the posterior probabilities of all the zero code words at the *l*th iteration), and apply E(*l*) measures to study their dynamical behaviours. Typical periodic and chaotic behaviours at low SNRs, periodic behaviours at waterfall region and two types of fixed points are discovered and systematically analyzed.**

Index Terms – Nonlinear dynamical system, turbo product codes, iterative decoding

I. INTRODUCTION

Turbo product codes (TPCs) built from two, three, or many smaller block codes are multidimensional array of block codes, which were also named as block turbo codes. In terms of the component (or constituent) codes, TPCs are categorized into Hamming product codes, extended Hamming product codes as well as single parity check product codes and so on [1]. The encoder of a two-dimensional turbo product code (TPC) consists of two sub-encoders, each of which is implemented by one block code. All component code may be identical or different. Due to the low implementation complexity compared to turbo codes, TPCs have been widely applied in satellite communication system [2], optical fiber communication systems [3],[4],[5], data memory systems [6] as well as the fourth generation mobile communication systems [1]. Specifically, turbo product codec has been used in satellite modems by Comtech EF Data corporation (a subsidiary of Comtech Telecommunications corporation) and Efficient Channel Coding Incorporated (ECC). In addition to the next generation direct broadcast satellite and other future satellite communication systems which include iPSTAR Broadband Satellite System, Intelsat satellite modems, satellite VSAT (Very Small Aperture Terminals) networks, ECC has also applied TPCs to emerging terrestrial systems, wireless networks (IEEE802.16), WiFi (IEEE802.11b), fiber optical links (ITU-G.709), and wireless public safety modems (TIA-902) [7]. For military applications, TPCs can be used in satellite-to-mobile systems as well as Phase III global broadcast service (GBS) terminals to take advantage of their improved performance and their flexible decoding.

The bifurcation and chaotic behaviours of turbo code decoders and LDPC code decoders have been reported recently in [8], [9], [10]. But the bifurcation and chaotic behaviours of turbo product code decoders have not been evaluated. The aim of this paper is to show a mathematical model of convergence of 2-D TPC decoder and analyse its performance, where TPC decoder can be regarded as a nonlinear feedback system.

Since turbo code decoders, LDPC code decoders and TPC decoders all use the iterative decoding algorithm, they have many commonalities. The key to the design of iterative algorithms is that every iteration produces probabilistic measures for all possible solutions. These measures are then fed back and used as input to the next iteration. Because every iteration accepts continuous-value input and produces continuous-value output, these algorithms are often referred to as soft decision algorithms. TPC decoders use concatenated row-column block code decoders which are different from turbo code decoder using convolutional codes.

The rest of this paper is organized as follows. In section II, the mathematic model of convergence of turbo product code decoder is summarized. In section III, the experimental results are given in detail. The conclusion is shown in Section IV.

II. THE MATHEMATICAL MODEL OF CONVERGENCE OF 2-D TURBO PRODUCT CODE DECODER

A. The Dynamical System of 2-D Turbo Product Code Decoder in AWGN Channel

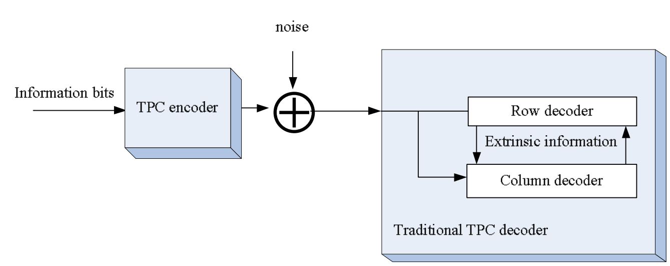


Fig. 1 A dynamical system of 2-D TPC Decoder

In Fig. 1, the 2-D TPC decoder has four parameters, which are signal-to-noise ratio (SNR), test pattern (*p*), weighting factor (α) as well as reliability factor (β).

From the dynamical system theory point of view, TPC decoder may be treated as a closed-loop dynamical system, where row decoder and column decoder act as the nonlinear constituent blocks in the corresponding open-loop system. In this system, we employ the Chase algorithm 2 into traditional TPC decoder[11].

B. The discription of E(l) measures

The performance of a TPC decoder is measured by the bit error rate (BER), which is a function of the aforementioned parameters, i.e.,

$$BER = f(SNR; p; \alpha ; \beta) \tag{1}$$

where α and β take 0.5 and 1 to get excellent performance , respectively. For simplification, p is set to be 2.

Without loss of generality, we assume the all-zero information sequence is transmitted as all "-1" sequence. Suppose a TPC (*n, k, d*) composed of two identical block codes, the length of each component code is *m*.

For an AWGN channel with the noise variance σ^2 , the reliability for the row decoder, or LLR (log-likelihood ratio) of bit position (*s; t*) is given by [11]

$$Q_{s,t}^{(l)} = \ln\frac{1-p_{s,t}^{(l)}(0)}{p_{s,t}^{(l)}(0)} = \frac{2}{\sigma^2}\lambda_{s,t}^{(l)} d_{s,t}^{(l)} \tag{2}$$

where *s*=1, 2, ..., *m*; *t*=1 , 2, ..., *m*; (*m* denotes TPC component code codeword length, e.g. m=15 for (15,11,3) Hamming code); *l* is iteration number; $p_{s,t}^{(l)}(0)$ is a posterior probability that the (*s, t*) *th* bit is "0" at the *l*th iteration; $d_{s,t}^{(l)} \in \{-1,1\}$; $\lambda_{s,t}^{(l)}$ is given by [11]

$$\lambda_{s,t}^{(l)} = \begin{cases} 0.25\left[\left|\mathrm{R}-\hat{\mathrm{D}}\right|^2 - \left|\mathrm{R}-\mathrm{D}\right|^2\right], \text{if a competing code word } \hat{\mathrm{D}} \text{ exits} \\ \beta, \quad \text{if no competing code word } \hat{\mathrm{D}} \text{ exits} \end{cases} \tag{3}$$

where $|\mathrm{X}-\mathrm{Y}|$ denotes the squared Euclidean distance between vectors **X** and **Y**.

R $=[r_{1,1}r_{1,2}...r_{1,m}r_{2,1}r_{2,2}...r_{2,m}...r_{m,1}r_{m,2}...r_{m,m}]$ is the received noisy sequence,

D $= [d_{1,1}^{(l)}d_{1,2}^{(l)}...d_{1,m}^{(l)}d_{2,1}^{(l)}d_{2,2}^{(l)}...d_{2,m}^{(l)}...d_{m,1}^{(l)}d_{m,2}^{(l)}...d_{m,m}^{(l)}]$ is the decided codeword after TPC decoding, and $\hat{\mathrm{D}} = [\hat{d}_{1,1}^{(l)}\hat{d}_{1,2}^{(l)}...\hat{d}_{1,m}^{(l)}\hat{d}_{2,1}^{(l)}\hat{d}_{2,2}^{(l)}...\hat{d}_{2,m}^{(l)}...\hat{d}_{m,1}^{(l)}\hat{d}_{m,2}^{(l)}...\hat{d}_{m,m}^{(l)}]$ (if it exists) is the most closest competing codeword to **R** among the 2^p candidate codewords with $\hat{d}_{s,t}^{(l)} \neq d_{s,t}^{(l)}$.

The probability in (2) can be expressed as

$$p_{s,t}^{(l)}(0) = \frac{1}{1+e^{\frac{2}{\sigma^2}\lambda_{s,t}^{(l)}d_{s,t}^{(l)}}} \tag{4}$$

When all-zero code bits transmitted are detected correctly, $\lambda_{s,t}^{(l)}$ is almost equal to 1, $d_{s,t}^{(l)} = -1$, $p_{s,t}^{(l)}(0)$ is close to 1 for all bit position (*s,t*).

For brevity, we apply the *E(l)* measures to study the bifurcation and chaotic behaviours of the decoder, where *E(l)* is defined as the mean square value of the posterior probabilities of $p_{s,t}^{(l)}(0)$ at the *l*th iteration, i.e.,

$$E(l) = \frac{1}{m^2}\sum_{s=1}^{m}\sum_{t=1}^{m}\left[p_{s,t}^{(l)}(0)\right]^2 \tag{5}$$

Clearly, substituting $p_{s,t}^{(l)}(0)=1$ in (5), we can obtain *E(l)*=1 .But the following simulation results show that the convergence of TPC decoder does not always induce *E(l)*=1. It is possible that in many cases the TPC decoder converges but we find *E(l)*<1. In other words, *E(l)*=1 is the sufficient but not necessary condition of the convergence of TPC decoder. The reason of *E(l)*<1 is that β is set in an empirical manner when the competing codeword is not available.

Substituting (5) in (6) results in

$$E(l) = \frac{1}{m^2}\sum_{s=1}^{m}\sum_{t=1}^{m}\frac{1}{\left(1+e^{\frac{2}{\sigma^2}\lambda_{s,t}^{(l)}d_{s,t}^{(l)}}\right)^2} \tag{6}$$

In this paper, we assume that the noise samples are represented by $\mathbb{Z} = (z_1, z_2, ...z_n)$. If the ratios of all consecutive sample values, i.e., $z_1/z_2, z_2/z_3, ..., z_{n-1}/z_n$ are fixed, we defined such noise samples as one noise realization. Different noise realizations correspond to different noise-ratio vectors. For TPC *(n,k)*^{2}, we get a noise realization by a given signal-to-noise ratio (SNR) with $\sigma^2 = 10^{-0.1SNR} = \frac{1}{n}\sum_{i=1}^{n}(z_i)^2$.

III. Experimental results

As an example of TPC $(15,11,3)^2$, we have performed many simulations by changing the parameter SNR from -2 dB to 6 dB with a typical noise realization.

We also observed that periodic, chaotic and convergence behaviours all exist as SNR changes. Simulation results show SNRs are divided into three different phases. First of all, periodicity and chaos occur alternatively at low SNR regions. Secondly, periodicity and the convergence with error bits occur at medium SNR regions. Finally, periodicity and the convergence with no error bits occur at high SNR regions. The Schemetic bifurcation diagram of the turbo product code decoder with a particular noise realization is shown in Fig. 2. The last 100 points of *E*(*l*) at each SNR are plotted.

A. *Bifurcation diagram*

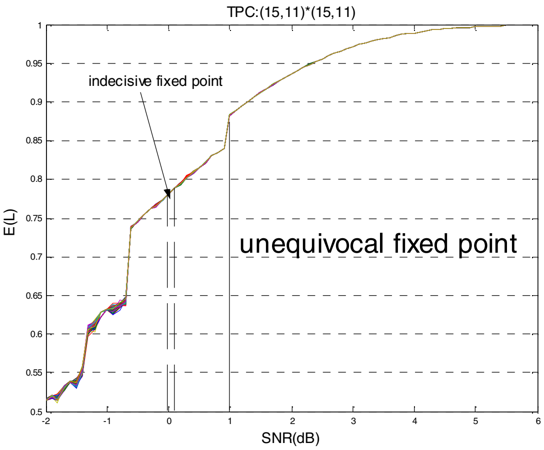


Fig. 2 Schemetic bifurcation diagram of the turbo product code decoder

In the turbo product code decoder, there are also two kinds of fixed points: one is the indecisive fixed point, in the region of 0dB to 0.1dB in our case; the other is the unequivocal fixed point which is represented by *E(l)*=1. However, in our system, when *E(l)* is equal to 0.88 (this is a simulation value corresponding to approximation value 0.85663 at SNR=1 dB), the unequivocal fixed points also occurs in TPC decoder as similar as turbo code decoder and LDPC code decoder. In Fig. 2, this transition corresponds to a narrow region of SNRs: from 0.1dB to 1 dB. This region is characterized by period cycle. These specifications are described in more details as follows.

B. *The periodic and chaotic behaviors*

For SNRs from –2.0 dB to –0.1 dB, periodic and chaotic behaviors occur alternatively. Table I describes their variation progresses with SNRs. It is clear that the two flip bifurcations occur at SNR=-1.45dB and SNR=-1.1dB as shown in Fig. 3 and Fig. 4 respectively. Fig. 5 demonstrates the occurrence of Neimark-Sacker bifurcation at SNR=-1.4dB. Fig. 6 discribes the chaotic behaviour at SNR=-1.65dB. The period-13 and period-12 cycles are shown in Fig. 7 and Fig. 8 respectively. Four period-6 cycles are shown in Fig. 9.

Our analysis indicates three routes to chaos: period-12 cycle, period-6 cycle and Neimark Sack bifurcation. Fig. 9 (a) shows that the TPC decoder spends nearly one thousand iterations via chaos before reaching the stable period-6 cycle. As the SNR increases, the number of iterations decreases, as can be seen in Fig. 9(c), the TPC decoder spends about 300 iterations via transient chaos to reach the stable period-6 cycle.

C. *The convergence with errorbits*

When the SNR is increased to 0dB, the phase trajectory of the TPC decoder converges to an indecisive fixed point.

This region is very narrow. Fig. 10 demonstrates the indecisive fixed point at SNR=0.1dB, where errors bits still exist.

TABLE I Typical examples of the periodic and chaotic behaviors at low SNR regions from –2dB to –0.1dB

Behaviours	SNR(dB)
Chaos	-1.65,-1.5,-1.3,-0.9
The flip bifurcation	-1.45,-1.1
Neimark Sack bifurcation	-1.4
Period-14 cycle	-1.91
Period-13 cycle	-2.0
Period-12 cycle	-1.9
Period-11 cycle	-1.99,-1.98,-1.97
Period-10 cycle	-0.3
Period-8 cycle	-1.95,-1.94,-1.93,-1.92
Period-6 cycle	-1.96,-1.6,-1.0,-0.5,-0.1

D. *The periodic behaviours and the convergence with no error bits*

When the SNR is further increased to 0.15 dB, the indecisive fixed point disappears and periodic behaviours also occur. Fig. 11 shows a bifurcation at SNR=0.8dB in the region from 0.15 dB to 1 dB.

At SNR=1 dB, the phase trajectory of the TPC decoder converges to a stable unequivocal fixed point as shown in Fig. Fig. 12, where all error bits are corrected.

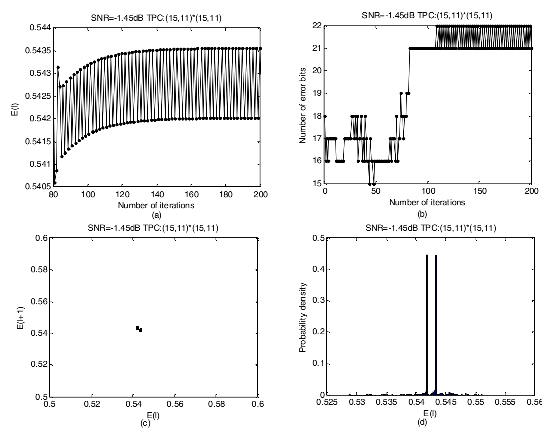


Fig. 3 The occurrence of two flip bifurcations at SNR=-1.45dB

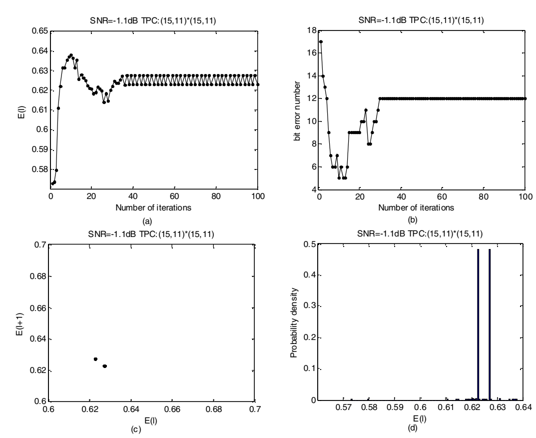


Fig. 4 The occurrence of two flip bifurcations at SNR=-1.1 dB

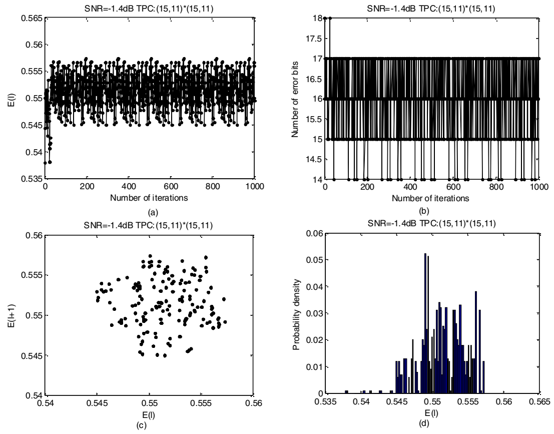


Fig. 5 The occurrence of Neimark-Sacker bifurcation at SNR=-1.4dB

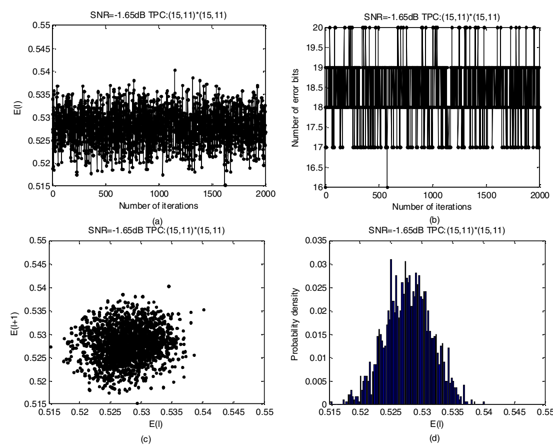


Fig. 6 The chaotic behaviours at SNR=-1.65dB

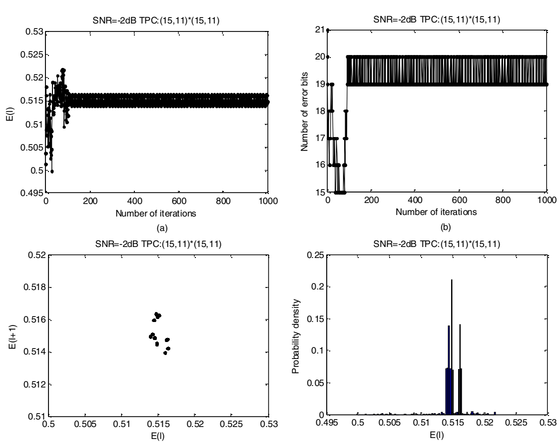


Fig. 7 The period-13 cycle at SNR=-2.0dB

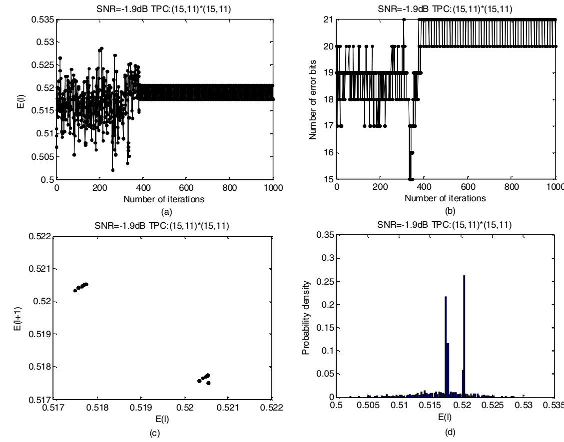


Fig. 8 The period-12 cycle at SNR=-1.9dB

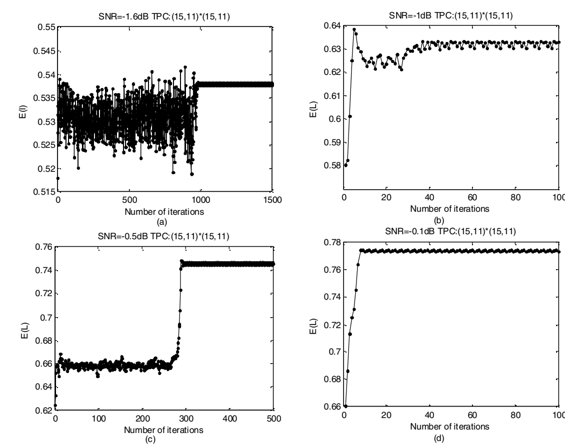


Fig. 9 The period-6 cycles

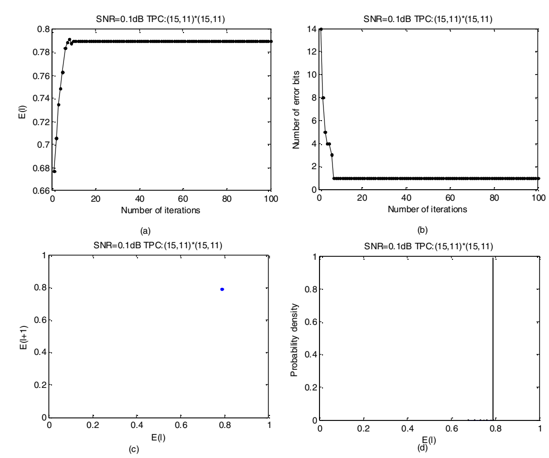


Fig. 10 The indecisive fixed point with error bits at SNR=0.1dB

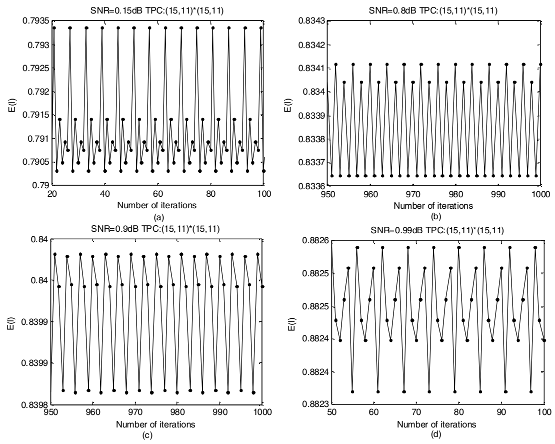
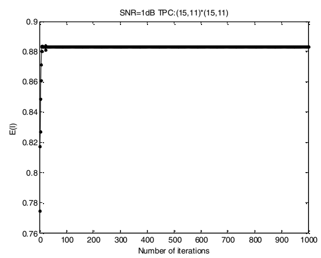
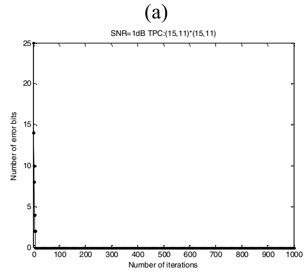


Fig. 11 A bifurcation in the interval [0.15,1]dB: (a) the period-6 cycle at SNR=0.15dB; (b)the period-4 cycle at SNR=0.8dB; (c)the period-6 cycle at SNR=0.9dB;(d) the period-6 cycle at SNR=0.99dB.



(a)



(b)

Fig. 12 The unequivocal fixed point with no an error bit at SNR=1dB

IV. Conclusion

In this paper, we investigated performance of a nonlinear dynamical system of two-dimensional turbo product code decoder. It is well known that the performance of is changed due to the SNR variation when other parameters are fixed. As a nonlinear system, two-dimensional turbo product code decoder exhibits a wide range of bifurcation and chaotic behaviour under some conditions. In this paper, the experimental evidence is provided to verify the predicted phenomena. The research about the domains of bifurcation and chaos in the parameter space is particularly important because the communication engineers must choose the parameter values in order to obtain the desired behavior. More over, the engineers will consciously avoid the bifurcation and chaos domains if they thoroughly understand when the nonlinear phenomena occur.

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